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Parallel Reductions in λ-Calculus

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The notion of parallel reduction is extracted from the Tait-Martin-Lof proof of Church-Rosser theorem (for β-reduction). We define parallel β-, η-, and βη-reduction by induction, and use them to give simple proofs of some fundamental theorems in λ-calculus; the postponement theorem of η-reduction (in βη-reduction), Church-Rosser theorem for βη-reduction, the normal reduction theorem for β-reduction, and some others.

1. Preliminaries

A λ-term is either x, λx.M (abstraction), or MN (application), where x is a variable and M, N are λ-terms. Unless otherwise stated, capital letters M,N,P,Q,R,... stand for arbitrary λ-terms, and x,y,z,u,v,... for arbitrary variables. We refer to [1] as the standard text; especially we use notations such as M[x:=N] (the substitution of N for free occurrences of x in M), M ≡ N (M is syntactically equal to N up to change of bound variables), $\underset{\beta}{\rightarrow}$, $\underset{\eta}{\rightarrow}$, $\underset{\beta\eta}{\rightarrow}$ (one-step β-, η-, βη-reductions, respectively), and $\underset{\beta}{\twoheadrightarrow}$, $\underset{\eta}{\twoheadrightarrow}$, $\underset{\beta\eta}{\twoheadrightarrow}$ (β-, η-, βη-reductions, respectively).

We define the parallel β-reduction $\underset{\beta}{\Rightarrow}$ inductively;

(β1) $x \underset{\beta}{\Rightarrow} x$,

(β2) $\lambda x.M \underset{\beta}{\Rightarrow} \lambda x.M'$ if $M \underset{\beta}{\Rightarrow} M'$,

(β3) $MN \underset{\beta}{\Rightarrow} M'N'$ if $M \underset{\beta}{\Rightarrow} M'$ and $N \underset{\beta}{\Rightarrow} N'$,

(β4) $(\lambda x.M)N \underset{\beta}{\Rightarrow} M'[x:=N']$ if $M \underset{\beta}{\Rightarrow} M'$ and $N \underset{\beta}{\Rightarrow} N'$.

Intuitively speaking, $M \underset{\beta}{\Rightarrow} M'$ means that M' is obtained from M by simultanuous reduction of some β-redexes existing in M. Clearly $M \underset{\beta}{\rightarrow} M'$ implies $M \underset{\beta}{\Rightarrow} M'$, and $M \underset{\beta}{\Rightarrow} M'$ implies $M \underset{\beta}{\twoheadrightarrow} M'$.

Thus $\underset{\beta}{\twoheadrightarrow}$ is the transitive closure of $\underset{\beta}{\Rightarrow}$ (but $\underset{\beta}{\Rightarrow}$ itself is not transitive). In [1], $M \underset{\beta}{\Rightarrow} M'$ is denoted by $M \underset{1}{\twoheadrightarrow} M'$.

Likewise, we define the parallel η-reduction $\underset{\eta}{\Rightarrow}$ by

(η1) $x \underset{\eta}{\Rightarrow} x$,

(η2) $\lambda x.M \underset{\eta}{\Rightarrow} \lambda x.M'$ if $M \underset{\eta}{\Rightarrow} M'$,

(η3) $\lambda v.Mv \underset{\eta}{\Rightarrow} M'$ if $M \underset{\eta}{\Rightarrow} M'$ and $v \notin FV(M)$,

(η4) $MN \underset{\eta}{\Rightarrow} M'N'$ if $M \underset{\eta}{\Rightarrow} M'$ and $N \underset{\eta}{\Rightarrow} N'$,

and the parallel $\beta\eta$-reduction $\underset{\beta\eta}{\Rightarrow}$ by

($\beta\eta$1) $x \underset{\beta\eta}{\Rightarrow} x$,

($\beta\eta$2) $\lambda x.M \underset{\beta\eta}{\Rightarrow} \lambda x.M'$ if $M \underset{\beta\eta}{\Rightarrow} M'$,

($\beta\eta$3) $\lambda v.Mv \underset{\beta\eta}{\Rightarrow} M'$ if $M \underset{\beta\eta}{\Rightarrow} M'$ and $v \notin FV(M)$,

($\beta\eta$4) $MN \underset{\beta\eta}{\Rightarrow} M'N'$ if $M \underset{\beta\eta}{\Rightarrow} M'$ and $N \underset{\beta\eta}{\Rightarrow} N'$,

($\beta\eta$5) $(\lambda x.M)N \underset{\beta\eta}{\Rightarrow} M'[x:=N']$ if $M \underset{\beta\eta}{\Rightarrow} M'$ and $N \underset{\beta\eta}{\Rightarrow} N'$.

(FV(M) stands for the set of free variables in M.) Intuitively, $M \underset{\eta}{\Rightarrow} M'$ ($M \underset{\beta\eta}{\Rightarrow} M'$, respectively) means that M' is obtained from M by simultanuous reduction of η-redexes (β-redexes and/or η-redexes) existing in M. As before we have

$$M \underset{\eta}{\rightarrow} M' \quad \Rightarrow \quad M \underset{\eta}{\Rightarrow} M' \quad \Rightarrow \quad M \underset{\eta}{\twoheadrightarrow} M',$$

$$M \underset{\beta\eta}{\rightarrow} M' \quad \Rightarrow \quad M \underset{\beta\eta}{\Rightarrow} M' \quad \Rightarrow \quad M \underset{\beta\eta}{\twoheadrightarrow} M'.$$

Therefore $\underset{\eta}{\twoheadrightarrow}$ ($\underset{\beta\eta}{\twoheadrightarrow}$, respectively) is the transitive closure of $\underset{\eta}{\Rightarrow}$ ($\underset{\beta\eta}{\Rightarrow}$).

These notions $\underset{\beta}{\Rightarrow}$, $\underset{\eta}{\Rightarrow}$, $\underset{\beta\eta}{\Rightarrow}$ are substitution closed in the sense;

$$M_i \underset{\beta}{\Rightarrow} M_i' \ (i=1,2) \quad \Rightarrow \quad M_1[x:=M_2] \underset{\beta}{\Rightarrow} M_1'[x:=M_2'],$$

$$M_i \underset{\eta}{\Rightarrow} M_i' \ (i=1,2) \quad \Rightarrow \quad M_1[x:=M_2] \underset{\eta}{\Rightarrow} M_1'[x:=M_2'],$$

$$M_i \underset{\beta\eta}{\Rightarrow} M_i' \ (i=1,2) \quad \Rightarrow \quad M_1[x:=M_2] \underset{\beta\eta}{\Rightarrow} M_1'[x:=M_2'].$$

See [1] lemma 3.2.4 for the inductive proof of the case $\underset{\beta}{\Rightarrow}$. Similar proofs apply to other cases, and are omitted.

In section 2, a short proof based on these notions is given for the theorem of postponement of η-reduction (in $\beta\eta$-reduction), together with some relations between $\underset{\beta}{\Rightarrow}$, $\underset{\eta}{\Rightarrow}$, $\underset{\beta\eta}{\Rightarrow}$. In section 3, we show that the idea of Tait-Martin-Lof proof of Church-Rosser theorem for $\underset{\beta}{\twoheadrightarrow}$ also applies to the case of $\underset{\beta\eta}{\twoheadrightarrow}$. In the final section, we present a direct proof of the normal reduction theorem for $\underset{\beta}{\twoheadrightarrow}$.

The essential difference of our proofs from previous ones is that the parallel reductions make simple inductive argument suffice to derive these theorems. In other words, by taking advantage of parallelism, one can avoid discussions of "residuals" and introduction of auxiliary terms other than λ-terms.

2. Relations Between Parallel Reductions

For any λ-term M, natural number $k \geq 0$, and variables v_1, v_2, ..., $v_k \notin FV(M)$, the λ-term $\lambda v_1.(\lambda v_2.(\ldots (\lambda v_k.Mv_k)\ldots)v_2)v_1$ is denoted by $(M)_k$. (In particular, $(M)_0 \equiv M$.) Then one can easily verify the following lemmas.

Lemma 2.1 Suppose $M \underset{\beta}{\Rightarrow} M'$, $N \underset{\beta}{\Rightarrow} N'$, and $k \geq 0$. Then

(1) $(\lambda x.M)_k \underset{\beta}{\Rightarrow} \lambda x.M'$,

(2) $(\lambda x.M)_k N \underset{\beta}{\Rightarrow} M'[x:=N']$,

(3) $(M)_k N \underset{\beta}{\Rightarrow} M'N'$,

(4) $(M)_{k+1} \underset{\beta}{\Rightarrow} (M')_1$

Lemma 2.2

(1) $M \underset{\eta}{\Rightarrow} x$ iff $M \equiv (x)_k$ for some $k \geq 0$.

(2) $M \underset{\eta}{\Rightarrow} N_1N_2$ iff $M \equiv (M_1M_2)_k$ for some $k \geq 0$ and M_i such that $M_i \underset{\eta}{\Rightarrow} N_i$ $(i=1,2)$.

(3) $M \underset{\eta}{\Rightarrow} \lambda x.N$ iff $M \equiv (\lambda x.M')_k$ for some $k \geq 0$ and M' such that $M' \underset{\eta}{\Rightarrow} N$.

Lemma 2.3 $M \underset{\eta}{\Rightarrow} P \underset{\beta}{\Rightarrow} N$ implies $M \underset{\beta}{\Rightarrow} P' \underset{\eta}{\Rightarrow} N$ for some P'.

(Proof) By induction on the structure of M. ::

Corollary 2.4 $M \underset{\beta\eta}{\twoheadrightarrow} N$ implies $M \underset{\beta}{\twoheadrightarrow} P \underset{\eta}{\twoheadrightarrow} N$ for some P.

By a similar inductive argument as lemma 2.3, one can verify the equivalence

$$M \underset{\beta\eta}{\Rightarrow} N \quad \text{iff} \quad M \underset{\eta}{\Rightarrow} P \underset{\beta}{\Rightarrow} N \text{ for some } P.$$

The converse of lemma 2.3 however does not hold. Indeed, $\lambda x.(\lambda y.yx)z \underset{\beta}{\Rightarrow} \lambda x.zx \underset{\eta}{\Rightarrow} z$, but not $\lambda x.(\lambda y.yx)z \underset{\beta\eta}{\Rightarrow} z$.

3. Church-Rosser Theorem for $\beta\eta$-Reduction

In this section we extend the simple proof due to Tait and Martin-Lof (see [1],[2]) of Church-Rosser theorem for $\underset{\beta}{\twoheadrightarrow}$ to the case of $\underset{\beta\eta}{\twoheadrightarrow}$.

Theorem 3.1 $M \underset{\beta\eta}{\twoheadrightarrow} M_i$ (i=1,2) implies $M_i \underset{\beta\eta}{\twoheadrightarrow} N$ (i=1,2) for some N.

(Proof) We define the λ-term $\tilde{M}$ for each M, as follows:

(1) If $M \equiv x$, then $\tilde{M} \equiv x$.

(2) If $M \equiv \lambda x.M_1$ and M is not a η-redex, then $\tilde{M} \equiv \lambda x.\tilde{M}_1$.

(3) If $M \equiv \lambda v.M_1 v$ and $v \notin FV(M_1)$, then $\tilde{M} \equiv \tilde{M}_1$.

(4) If $M \equiv M_1 M_2$ and M is not a β-redex, then $\tilde{M} \equiv \tilde{M}_1\tilde{M}_2$.

(5) If $M \equiv (\lambda x.M_1)M_2$, then $\tilde{M} \equiv \tilde{M}_1[x:=\tilde{M}_2]$.

Then one can verify by induction on the structure of M that $M \underset{\beta\eta}{\rightrightarrows} N$ implies $N \underset{\beta\eta}{\rightrightarrows} \tilde{M}$ for any M and N. It means that $\underset{\beta\eta}{\rightrightarrows}$ satisfies the diamond property;

$M \underset{\beta\eta}{\rightrightarrows} M_i$ (i=1,2) implies $M_i \underset{\beta\eta}{\rightrightarrows} N$ (i=1,2) for some N,

from which the theorem immediately follows since $\underset{\beta\eta}{\twoheadrightarrow}$ is the transitive closure of $\underset{\beta\eta}{\rightrightarrows}$. ::

The foregoing proof also shows:

Theorem 3.2 Let $\tilde{M}^0 \equiv M$ and $\tilde{M}^{n+1} \equiv \tilde{P}$ for $P \equiv \tilde{M}^n$ $(n \geq 0)$. Then $M \underset{\beta\eta}{\twoheadrightarrow} N$ implies $N \underset{\beta\eta}{\twoheadrightarrow} \tilde{M}^n$ for some $n \geq 0$.

4. Normal Reduction Theorem for β-Reduction

The aim of this section is to give a direct proof of normal reduction theorem for $\underset{\beta}{\twoheadrightarrow}$; if M has a β-normal form N, then N can be obtained from M by the leftmost β-reduction.

We use notations $\xrightarrow[\beta]{h}\!\!\!\!\!\rightarrow$, $\xrightarrow[\beta]{i}\!\!\!\!\!\rightarrow$, and $\xrightarrow[\beta]{l}\!\!\!\!\!\rightarrow$ for head β-reduction, internal β-reduction, and leftmost β-reduction, respectively. One-step head β-reduction is denoted by $\xrightarrow[\beta]{h}$.

Lemma 4.1

(1) $M \xrightarrow[\beta]{h} N$ implies $\lambda x.M \xrightarrow[\beta]{h} \lambda x.N$.

(2) $M \xrightarrow[\beta]{h} N$ implies $M[x:=P] \xrightarrow[\beta]{h} N[x:=P]$.

(3) $M \xrightarrow[\beta]{h} N$ implies $MP \xrightarrow[\beta]{h} NP$, unless M is an abstraction.

We define the parallel internal ß-reduction $\overset{i}{\underset{\beta}{\Rightarrow}}$ inductively;

(iß1) $x \overset{i}{\underset{\beta}{\Rightarrow}} x$,

(iß2) $\lambda x.M \overset{i}{\underset{\beta}{\Rightarrow}} \lambda x.M'$ if $M \overset{i}{\underset{\beta}{\Rightarrow}} M'$,

(iß3) $MN \overset{i}{\underset{\beta}{\Rightarrow}} M'N'$ if $M \overset{i}{\underset{\beta}{\Rightarrow}} M'$ and $N \underset{\beta}{\Rightarrow} N'$,

(iß4) $(\lambda x.M)N \overset{i}{\underset{\beta}{\Rightarrow}} (\lambda x.M')N'$ if $M \underset{\beta}{\Rightarrow} M'$ and $N \underset{\beta}{\Rightarrow} N'$.

Clearly $P \xrightarrow[\beta]{i} Q$ implies $P \overset{i}{\underset{\beta}{\Rightarrow}} Q$, which in turn implies $P \overset{i}{\underset{\beta}{\twoheadrightarrow}} Q$. More precisely, $P \overset{i}{\underset{\beta}{\Rightarrow}} Q$ holds iff either

(1) $P \equiv \lambda\vec{y}.xP_1P_2\ldots P_n$ and $Q \equiv \lambda\vec{y}.xQ_1Q_2\ldots Q_n$ with $n \geq 0$ and $P_j \underset{\beta}{\Rightarrow} Q_j$ $(j=1,2,\ldots,n)$, or

(2) $P \equiv \lambda\vec{y}.(\lambda x.P_0)P_1P_2\ldots P_n$ and $Q \equiv \lambda\vec{y}.(\lambda x.Q_0)Q_1Q_2\ldots Q_n$ with $n \geq 1$ and $P_j \underset{\beta}{\Rightarrow} Q_j$ $(j=0,1,\ldots,n)$

for a sequence $\vec{y}$ of variables, a variable x, and λ-terms P_j, Q_j.

The key lemma in our proof of the normal reduction theorem is the following.

Lemma 4.2 $M \underset{\beta}{\Rightarrow} N$ implies $M \overset{h}{\underset{\beta}{\twoheadrightarrow}} P \overset{i}{\underset{\beta}{\Rightarrow}} N$ for some P.

(Proof) We prove a stronger statement;

$$M \underset{\beta}{\Rightarrow} N \text{ implies } M \xrightarrow[\beta]{h} M' \xrightarrow[\beta]{h} M'' \xrightarrow[\beta]{h} \cdots \xrightarrow[\beta]{h} M^{(k)} \overset{i}{\underset{\beta}{\Rightarrow}} N$$
$$\text{for some } k \geq 0 \text{ and } M^{(j)} \underset{\beta}{\Rightarrow} N \quad (j=1,\ldots,k)$$

by induction on the structure of M. (Here we identify M with $M^{(0)}$, M' with $M^{(1)}$, etc.) There are two possibilities.

(1) If $M \equiv \lambda\vec{y}.xM_1M_2...M_n$, then $N \equiv \lambda\vec{y}.xN_1N_2...N_n$ where $M_j \underset{\beta}{\Rightarrow} N_j$ $(j=1,...,n)$. In this case clearly $M \overset{i}{\underset{\beta}{\Rightarrow}} N$.

(2) If $M \equiv \lambda\vec{y}.(\lambda x.M_0)M_1M_2...M_n$ with $n \geq 1$, then either (2.1) $N \equiv \lambda\vec{y}.(\lambda x.N_0)N_1N_2...N_n$ or (2.2) $N \equiv \lambda\vec{y}.(N_0[x:= N_1])\, N_2...N_n$ where $M_j \underset{\beta}{\Rightarrow} N_j$ $(j=0,1,...,n)$. In case (2.1), by definition $M \overset{i}{\underset{\beta}{\Rightarrow}} N$. We prove the case (2.2), by assuming $\vec{y}$ is empty. (It does not lose generality, because of lemma 4.1(1) and definition (iβ2).) In this case, $M \equiv (\lambda x.M_0)M_1M_2...M_n \overset{h}{\underset{\beta}{\rightarrow}} (M_0[x:= M_1])M_2...M_n \underset{\beta}{\Rightarrow} (N_0[x:= N_1])N_2...N_n \equiv N$. Since $M_j \underset{\beta}{\Rightarrow} N_j$ and M_j's are subterms of M, by inductive hypothesis we know $M_j \overset{h}{\underset{\beta}{\rightarrow}} M_j' \overset{h}{\underset{\beta}{\rightarrow}} M_j'' \overset{h}{\underset{\beta}{\rightarrow}} \cdots \overset{h}{\underset{\beta}{\rightarrow}} M_j^{(k_j)} \overset{i}{\underset{\beta}{\Rightarrow}} N_j$ for some $M_j^{(k)} \underset{\beta}{\Rightarrow} N_j$ $(k=0,1,...,k_j)$, and $k_j \geq 0$ $(j = 0,1,...,n)$. Then from lemma 4.1(2) and lemma 4.4 below, we get

$$\begin{aligned} M_0[x:=M_1] &\overset{h}{\underset{\beta}{\rightarrow}} M_0'[x:=M_1] \overset{h}{\underset{\beta}{\rightarrow}} M_0''[x:=M_1] \overset{h}{\underset{\beta}{\rightarrow}} \cdots \\ &\overset{h}{\underset{\beta}{\rightarrow}} M_0^{(k_0)}[x:=M_1] \overset{h}{\underset{\beta}{\rightarrow}} M_0^{(k_0)}[x:=M_1'] \overset{h}{\underset{\beta}{\rightarrow}} M_0^{(k_0)}[x:=M_1''] \\ &\overset{h}{\underset{\beta}{\rightarrow}} \cdots \overset{h}{\underset{\beta}{\rightarrow}} M_0^{(k_0)}[x:=M_1^{(k)}] \overset{i}{\underset{\beta}{\Rightarrow}} N_0[x:=N_1] \qquad (*) \end{aligned}$$

for some $k \leq k_1$. Since $M_0^{(j)} \underset{\beta}{\Rightarrow} N_0$ $(j=0,1,...,k_0)$ and $M_1^{(j)} \underset{\beta}{\Rightarrow} N_1$ $(j=0,1,...,k)$, we have

$$\begin{aligned} &M_0^{(j)}[x:=M_1] \underset{\beta}{\Rightarrow} N_0[x:=N_1] \qquad (j=0,1,...,k_0), \text{ and} \\ &M_0^{(k_0)}[x:=M_1^{(j)}] \underset{\beta}{\Rightarrow} N_0[x:=N_1] \qquad (j=0,1,...,k). \end{aligned}$$

Let $M_0[x:=M_1] \equiv P \overset{h}{\underset{\beta}{\rightarrow}} P' \overset{h}{\underset{\beta}{\rightarrow}} P'' \overset{h}{\underset{\beta}{\rightarrow}} \cdots \overset{h}{\underset{\beta}{\rightarrow}} P^{(k_0+k)} \overset{i}{\underset{\beta}{\Rightarrow}} Q \equiv N_0[x:=N_1]$ stand for the reduction sequence (*). Then by applying lemma 4.3 below to (*) n times, we get

$$\begin{aligned} M &\overset{h}{\underset{\beta}{\rightarrow}} PM_2M_3...M_n \overset{h}{\underset{\beta}{\rightarrow}} P'M_2M_3...M_n \overset{h}{\underset{\beta}{\rightarrow}} \cdots \\ &\cdots \overset{h}{\underset{\beta}{\rightarrow}} P^{(p)}M_2M_3...M_n \overset{i}{\underset{\beta}{\Rightarrow}} QN_2N_3...N_n \equiv N, \text{ and} \\ &P^{(j)}M_2M_3...M_n \underset{\beta}{\Rightarrow} N \quad (j=0,1,...,p) \end{aligned}$$

for some $p \leq k_0+k$. ::

Lemma 4.3 If $M \xrightarrow[\beta]{h} M' \xrightarrow[\beta]{h} M'' \xrightarrow[\beta]{h} \ldots \xrightarrow[\beta]{h} M^{(k)} \xRightarrow[\beta]{i} N$ with $M^{(j)} \xRightarrow[\beta]{} N$ $(j=0,1,\ldots,k)$, and $P \xRightarrow[\beta]{} Q$, then for some $m \le k$

$$MP \xrightarrow[\beta]{h} M'P \xrightarrow[\beta]{h} M''P \xrightarrow[\beta]{h} \ldots \xrightarrow[\beta]{h} M^{(m)}P \xRightarrow[\beta]{i} NQ, \text{ and}$$

$$M^{(j)}P \xRightarrow[\beta]{} NQ \ (j=0,1,\ldots,m).$$

(Proof) If there exist abstrasctions in M, M', M'',...,$M^{(k)}$, let $M^{(m)}$ be the first one in the sequence. Then by lemma 4.1(3) and definition (iβ4)

$$MP \xrightarrow[\beta]{h} M'P \xrightarrow[\beta]{h} M''P \xrightarrow[\beta]{h} \ldots \xrightarrow[\beta]{h} M^{(m)}P \xRightarrow[\beta]{i} NQ$$

since $M^{(m)} \xRightarrow[\beta]{} N$ implies that N is also an abstraction. On the other hand, if there is no abstraction in M, M', ..., $M^{(k)}$, then for $m = k$ we have

$$MP \xrightarrow[\beta]{h} M'P \xrightarrow[\beta]{h} M''P \xrightarrow[\beta]{h} \ldots \xrightarrow[\beta]{h} M^{(m)}P \xRightarrow[\beta]{i} NQ$$

by lemma 4.1(3) and definition (iβ3). In either case, clearly $M^{(j)}P \xRightarrow[\beta]{} NQ$ $(j=0,1,\ldots,m)$. ::

Lemma 4.4 If $M \xrightarrow[\beta]{h} M' \xrightarrow[\beta]{h} M'' \xrightarrow[\beta]{h} \ldots \xrightarrow[\beta]{h} M^{(k)} \xRightarrow[\beta]{i} N$ with $M^{(j)} \xRightarrow[\beta]{} N$ $(j=0,1,\ldots,k)$, and $P \xRightarrow[\beta]{i} Q$, then for some $m \le k$

$$P[x:=M] \xrightarrow[\beta]{h} P[x:=M'] \xrightarrow[\beta]{h} P[x:=M''] \xrightarrow[\beta]{h} \ldots$$
$$\ldots \xrightarrow[\beta]{h} P[x:=M^{(m)}] \xRightarrow[\beta]{i} Q[x:=N], \text{ and}$$

$$P[x:=M^{(j)}] \xRightarrow[\beta]{} Q[x:=N] \quad (j=0,1,\ldots,m).$$

(Proof) Since $P \xRightarrow[\beta]{i} Q$, we have either

(1) $P \equiv \lambda\vec{y}.zP_1P_2\ldots P_n$ and $Q \equiv \lambda\vec{y}.zQ_1Q_2\ldots Q_n$ with $n \ge 0$ and $P_j \xRightarrow[\beta]{} Q_j$ $(j=1,\ldots,n)$, or

(2) $P \equiv \lambda\vec{y}.(\lambda z.P_0)P_1P_2\ldots P_n$ and $Q \equiv \lambda\vec{y}.(\lambda z.Q_0)Q_1Q_2\ldots Q_n$ with $n \ge 1$ and $P_j \xRightarrow[\beta]{} Q_j$ $(j=0,1,\ldots,n)$.

Because of lemma 4.1(1) and definition (iβ2), it suffices to consider the case where $\vec{y}$ is empty. Let $P_j' \equiv P_j[x:=M]$ and $Q_j' \equiv Q_j[x:=N]$ $(j=0,1,\dots,n)$. If $z \equiv x$ in case (1), then by lemma 4.3

$$P[x:=M] \equiv MP_1'P_2'\dots P_n' \xrightarrow[\beta]{h} M'P_1'P_2'\dots P_n' \xrightarrow[\beta]{h} \dots \xrightarrow[\beta]{h}$$
$$M^{(m)}P_1'P_2'\dots P_n' \overset{i}{\underset{\beta}{\Rightarrow}} NQ_1'Q_2'\dots Q_n' \equiv Q[x:=N], \text{ and}$$
$$M^{(j)}P_1'P_2'\dots P_n' \underset{\beta}{\Rightarrow} NQ_1'Q_2'\dots Q_n' \quad (j=0,1,\dots,m)$$

for some $m \le k$. If in (1) z is different from x, then clearly $P[x:=M] \equiv zP_1'P_2'\dots P_n' \overset{i}{\underset{\beta}{\Rightarrow}} zQ_1'Q_2'\dots Q_n' \equiv Q[x:= N]$. In case (2), $P[x:=M] \equiv (\lambda z.P_0')P_1'P_2'\dots P_n' \overset{i}{\underset{\beta}{\Rightarrow}} (\lambda z.Q_0')Q_1'Q_2'\dots Q_n' \equiv Q[x:=N]$. ::

The proof of lemma 4.2 is now completed. It says that $M \underset{\beta}{\Rightarrow} N$ implies $M \overset{h}{\underset{\beta}{\twoheadrightarrow}} P \overset{i}{\underset{\beta}{\twoheadrightarrow}} N$ for some P. Next we show that the same holds true under a weaker condition $M \underset{\beta}{\twoheadrightarrow} N$.

Lemma 4.5 $M \overset{i}{\underset{\beta}{\Rightarrow}} P \xrightarrow[\beta]{h} N$ implies $M \overset{h}{\underset{\beta}{\twoheadrightarrow}} Q \overset{i}{\underset{\beta}{\Rightarrow}} N$ for some Q.

(Proof) Since $P \xrightarrow[\beta]{h} N$, we have $P \equiv \lambda\vec{y}.(\lambda x.P_0)P_1P_2\dots P_n$ and $N \equiv \lambda\vec{y}.(P_0[x:=P_1])P_2\dots P_n$ for some $n \ge 1$ and $\vec{y}$, x, P_0, P_1, ..., P_n. Next from $M \overset{i}{\underset{\beta}{\Rightarrow}} P$, we know that $M \equiv \lambda\vec{y}.(\lambda x.M_0)M_1M_2\dots M_n$ with some $M_i \underset{\beta}{\Rightarrow} P_i$ $(j=0,1,\dots,n)$. This implies $M \xrightarrow[\beta]{h} \lambda\vec{y}.(M_0[x:=M_1])M_2\dots M_n \underset{\beta}{\Rightarrow} \lambda\vec{y}.(P_0[x:=P_1])P_2\dots P_n \equiv N$, which together with lemma 4.2 shows the lemma. ::

Corollary 4.6 $M \underset{\beta}{\twoheadrightarrow} N$ implies $M \overset{h}{\underset{\beta}{\twoheadrightarrow}} P \overset{i}{\underset{\beta}{\twoheadrightarrow}} N$ for some P.

(Proof) Recall that $\underset{\beta}{\twoheadrightarrow}$ ($\overset{i}{\underset{\beta}{\twoheadrightarrow}}$, resp.) is the transitive closure of $\underset{\beta}{\Rightarrow}$ (of $\overset{i}{\underset{\beta}{\Rightarrow}}$), and apply lemmas 4.2 and 4.5. ::

From corollary 4.6, one can obtain the normal reduction theorem as in [1].

Theorem 4.7 If M has a ß-normal form N, then M $\xrightarrow[\beta]{1}\!\!\!\rightarrow$ N.

Similarly we can prove from corollary 4.6 the standardization theorm ([1] theorem 11.4.7).

References

[1] H.P.Barendregt, The Lambda Calculus (North-Holland 1984)

[2] J.J.Levy, An algebraic interpretation of the λ-ß-K calculus and a labelled λ-calculus (in Lecture Notes in Computer Science 37, Springer-Verlag 1975) pp 147-165.